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1994 J. Phys. A: Math. Gen. 27 L163

(http://iopscience.iop.org/0305-4470/27/6/003)

J. Phys. A: Math. Gen. 27 (1994) L163–L169. Printed in the UK

LETTER TO THE EDITOR

Scaling function in the universality class of two-dimensional Ising model

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Received 21 January 1994

Abstract. The fermionic model is considered equivalent in the critical region to the two-dimensional Ising model in homogeneous magnetic field. The ground state of the fermionic Hamiltonian is obtained from the variational principle of the quantum mechanics, where probe states are taken in the BCS-like form. In this approximation the explicit representation of the scaling function is derived which is characterized by the correct values of critical indexes and the susceptibility critical amplitudes ratio $C_+/C_- = 12\pi$.

Fifty years ago Onsager [1] gave his celebrated exact solution of the two-dimensional Ising model for the square lattice in zero magnetic field. Twenty years later Schultz, Mattis and Lieb in a remarkable paper [2] proved the equivalence of this model with a certain BCS-like one-dimensional fermionic model and obtained the solution in a very elegant and simplified manner. Their fermionic treatment turned out to be very fruitful and stimulated greatly the further progress in understanding of the critical properties of the Ising model (see e.g. [3, 4]). Here we study a simple fermionic model equivalent, in the critical region, to the two-dimensional Ising model in uniform magnetic field.

Consider the fermionic model defined by the Hamiltonian

$$R = R_0 + V \tag{1a}$$

$$R_0 = \int_0^L \mathrm{d}x \left\{ \Omega_0 \psi^+ \psi + s \frac{\mathrm{d}\psi^+}{\mathrm{d}x}\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{i\Gamma}{2}\left(\frac{\mathrm{d}\psi^+}{\mathrm{d}x}\psi^+ + \frac{\mathrm{d}\psi}{\mathrm{d}x}\psi\right)\right\} \tag{1b}$$

$$V = -H \int_0^L \mathrm{d}x \sigma(x). \tag{1c}$$

Here L is the systems length in the x-direction, $\psi^+(x)$ and $\psi(x)$ are the Fermi operators creating/annihilating a domain wall at a point x. More precisely, fermion trajectories $x(\tau)$ in the (x, τ)-plane correspond to the domain wall lines in the (x, y)-plane (see e.g. [5]). Such a wall separates regions with Ising spin values $\sigma(x, y) = +1$ and $\sigma(x, y) = -1$. Ising spin operator $\sigma(x)$ is to be expressed in terms of the Fermi operators $\psi^+(x)$ and $\psi(x)$ as it is described below. Parameter Ω_0 is proportional to the reduced temperature: $\Omega_0 = -\varkappa t$, $\varkappa > 0$, $t = (T - T_c)/T_c$. The last term in the right-hand side of (1b) permits us to describe finite-size domains. As it is implied by the fermion analogy, the free energy of two-dimensional classical system is proportional to the ground-state energy E of the quantum one-dimensional Hamiltonian (1).

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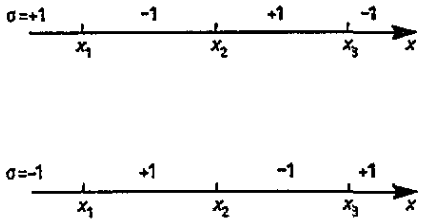

Figure 1. Two spin configurations corresponding to the state (2).

Hamiltonian (1) operates in the Fock-space with basis formed by vectors

$$\psi^+(x_1)\psi^+(x_2)\psi^+(x_3)\ldots|0\rangle. \tag{2}$$

Two spin configurations shown in figures 1(*a*) and 1(*b*) correspond to such a state. These configurations are transformed one into another by overturn of all the spins. Hence, operator $\sigma(x)$ is not well defined in our fermion representation. However, one can easily define the product of the two spin operators

$$\sigma(x'')\sigma(x') = \exp[\mathrm{i}\pi N(x'', x')] \tag{3}$$

where $N(x'', x')$ is the operator of the number of domain walls between points x'' and x':

$$N(x'', x') = \int_{x'}^{x''} \mathrm{d}x\, \psi^+(x)\psi(x). \tag{4}$$

Really, if $N(x'', x')$ is even, spins $\sigma(x')$ and $\sigma(x'')$ are the same, if $N(x'', x')$ is odd, spins $\sigma(x')$ and $\sigma(x'')$ have opposite direction. Relation (3) leads to the needed formula for the spontaneous magnetization $M(\Phi)$ in a translationary invariant state $|\Phi\rangle$:

$$M(\Phi) \equiv \langle\sigma\rangle_\Phi \equiv \left\{ \lim_{x''-x'\to\infty} \langle\Phi|\exp[\mathrm{i}\pi N(x'', x')]|\Phi\rangle / \langle\Phi|\Phi\rangle \right\}^{1/2}. \tag{5}$$

The model (1) in zero field $H = 0$ was considered in [6], where it was shown that it is of the Ising universality class†. In this case the Hamiltonian is diagonalized by the Bogoliubov transformation and the ground state has the BCS-form:

$$|\Phi\rangle = \exp\left\{ \frac{1}{2} \int \mathrm{d}x_1\, \mathrm{d}x_2 \psi^+(x_1)\psi^+(x_2)G(x_1 - x_2) \right\} |0\rangle \tag{6a}$$

where

$$G(x) = \int \frac{\mathrm{d}p}{2\pi} \Lambda_0(p) \exp(\mathrm{i}px) \tag{6b}$$

$$\Lambda_0(p) = \frac{\Omega_0 + sp^2 - \sqrt{(\Omega_0 + sp^2)^2 + (\Gamma p)^2}}{\Gamma p}. \tag{6c}$$

† For analogous discrete model and the same representation of Ising spin operators this result was obtained earlier by Bohr [7].

If $H \neq 0$, one cannot diagonalize Hamiltonian (1). Instead let us use the variational principle of the quantum mechanics: the ground-state energy E of Hamiltonian (1) is equal to the minimum value of the functional $f(\Phi)$ given by

$$f(\Phi) = \left\{ \frac{\langle\Phi|R_0|\Phi\rangle}{\langle\Phi|\Phi\rangle} - LHM(\Phi) \right\} \tag{7a}$$

$$E = \min_{\Phi} f(\Phi) \tag{7b}$$

where the state $|\Phi\rangle$ is assumed translationary invariant. Now we adopt the basic approximation contracting the class of varying states in (7*b*) to the BCS-like states only. Thus, we suppose, that in the critical region (for small values of t and H) the ground state can be fairly approximated by formulae (6*a*) and (6*b*) where, however, the function $\Lambda_0(p)$ is replaced by the unknown varying function $\Lambda(p)$. We adopt that this function is of the Morse class represented by the function $\Lambda_0(p)$ in the ordered phase $\Omega_0 > 0$ (see figure 2). This is an odd function going to zero as $x \to \pm\infty$ and having a single minimum in the half-line $p > 0$ with the value $-\Lambda_{\mathrm{m}} < 0$.

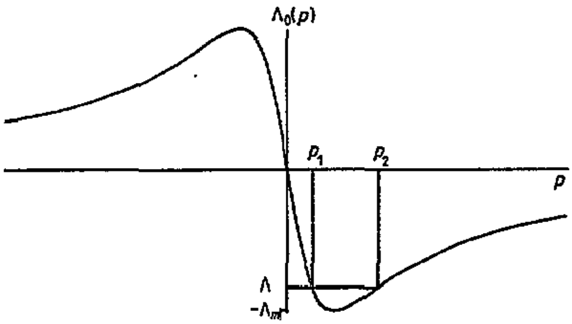

Figure 2. The function $\Lambda_0(p)$ in the low-temperature phase $\Omega_0 > 0$.

Both terms in (7*a*) can be exactly calculated for such a state. This is obvious for the first one, since R_0 is quadratic in ψ and ψ^+. For the magnetization (5) in the BCS-state described above, the following representation is valid:

$$M = (1 + \Lambda_{\mathrm{m}}^2)^{-1/2} \exp\left\{ \int_{-\Lambda_{\mathrm{m}}}^{0} \frac{\Lambda \mathrm{d}\Lambda}{\pi(1+\Lambda^2)} [\alpha_2(\Lambda) - \alpha_1(\Lambda)] \right\} \tag{8a}$$

where

$$\alpha_j(\Lambda) = \mathrm{Im}\left\{ \frac{1}{i\pi} \int_{-\infty+i0}^{\infty+i0} \frac{\mathrm{d}q}{q + p_j(\Lambda)} \ln \frac{\Lambda + \Lambda(q)}{\Lambda - \Lambda(q)} \right\} \tag{8b}$$

and $p_j(\Lambda)$ are the solutions of the equation $\Lambda(p)\mid_{p=p_j} = \Lambda$, $0 < p_1 < p_2 < \infty$. The integration path in (8*b*) is slightly shifted into the upper half-plane.

Thus, we have the integral representation for the functional $f(\Lambda) \equiv f(\Phi(\Lambda))$. Varying it with respect to $\Lambda(p)$ one obtains the nonlinear singular equation defining $\Lambda(p)$:

$$(\Omega_0 + sp^2)\sin 2\varphi(p) + \Gamma p \cos 2\varphi(p) = -\frac{2MH}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{(q+p)} \frac{\mathrm{d}\varphi(q)}{\mathrm{d}q} \tag{9}$$

where $\varphi(p)$ is the angle of the Bogoliubov transformation: $\Lambda(p) = \tan[\varphi(p)]$. Integration in (9) and in the further equation (12) is understood in the sense of the Cauchy principal value. Substituting the solution of the above equation into the relation (8) we obtain the equation of states in the adopted approximation. In the critical region it can be written in the form of the well known scaling relation proposed by Griffiths [9]:

$$H = \mathrm{const} \cdot M^{15} h(x_s). \tag{10}$$

Here x_s is the familiar scaling variable

$$x_s = \frac{t}{M^8} \frac{4\varkappa s}{\Gamma^2} \tag{11}$$

that lies in the interval $-1 < x_s < \infty$ for equilibrium states. The scaling function $h(x_s)$ is constructed as follows.

Denote by $\varphi(p; y_s)$ the solution of the nonlinear singular equation

$$\mathrm{sign}(t) \cdot \sin 2\varphi(p) - p \cdot \cos 2\varphi(p) = \frac{y_s}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{(q+p)} \frac{\mathrm{d}\varphi(q)}{\mathrm{d}q} \tag{12}$$

with boundary conditions $\varphi(\infty; y_s) = -\pi/4$, $\varphi(-\infty; y_s) = \pi/4$. The parameter y_s is given by

$$y_s = \frac{2MH\Gamma}{\Omega_0^2}. \tag{13}$$

Scaling variable x_s is related to y_s by

$$x_s = \mathrm{sign}(t) \left(\frac{1+\Lambda_m^2}{2}\right)^4 \exp\left\{ \int_{-\Lambda_m}^{0} \frac{8\Lambda \mathrm{d}\Lambda}{\pi(1+\Lambda^2)} [\alpha_1(\Lambda) - \alpha(\Lambda)] \right\} \tag{14}$$

where Λ_m is the maximum value of the function $\Lambda(p)$ given by

$$\Lambda(p) = \tan \varphi(p; y_s). \tag{15}$$

The function $\alpha(\Lambda)$ is defined via

$$a(\Lambda) = \begin{cases} \alpha_0(\Lambda) & \text{if } -1 < \Lambda < 0 \\ \alpha_2(\Lambda) & \text{if } -\Lambda_m < \Lambda < -1. \end{cases} \tag{16}$$

The functions $\alpha_j(\Lambda)$ ($j = 0, 1, 2$) are given by the relation (8*b*), where for $j = 1, 2$ $\Lambda(p)$ is taken in the form (15), whereas for $j = 0$ $\Lambda(p)$ is the rescaled zero-field function in the low-temperature phase (compare with (6*c*)):

$$\Lambda(p) = \frac{1 - \sqrt{p^2+1}}{p}. \tag{17}$$

Though the integral in (8*b*) defining $\alpha_j(\Lambda)$ has logarithmic divergence at large q, diverging terms cancel each other in the square brackets in the right-hand side of (14).

In deriving (14) we divided the magnetization $M(\Omega_0, H)$ given by (8*a*) by the similar expression for the zero-field magnetization in the ordered phase $M_0 \equiv M(|\Omega_0|, 0)$ and have taken into account that the latter can be reduced to

$$M_0 \cong (4\varkappa s\Gamma^{-2}|t|)^{1/8} \tag{18}$$

in the critical region [6].

Thus, the relation (14) defines the function $x_s(y_s)$. The inverse function $y_s(x_s)$ with prefactor x_s^2 gives the desired scaling function

$$h(x_s) = x_s^2 y_s(x_s). \tag{19}$$

Now let us discuss some consequences following immediately from the obtained results.

(1) First, it is clear from the above consideration, that the derived equation of states is characterized to be correct Ising critical exponents $\beta = 1/8$ and $\delta = 15$. Further, certain more or less straightforward perturbation theory calculations lead to the critical asymptote of the magnetic susceptibility $\chi = \partial M/\partial H$ in the high- and low-temperature phases:

$$\chi_\pm = C_\pm |t|^{-7/4} \tag{20}$$

with

$$C_+ = 2\frac{(4s\Gamma^2)^{1/4}}{\varkappa^{7/4}} \qquad C_- = \frac{(4s\Gamma^2)^{1/4}}{6\pi\varkappa^{7/4}}.$$

Thus, we obtain the expected values of exponents γ and γ' ($\gamma = \gamma' = 7/4$) as well as the universal ratio of critical amplitudes:

$$C_+/C_- = 12\pi = 37.699\,111\,843. \tag{21}$$

The latter is very close to the value 37.6936 given by Barouch, McCoy and Wu [9].

(2) Somewhat less precise, and yet more simple expression approximating the scaling function $h(x_s)$ was derived in the paper [10]. It leads to the value $C_+/C_- = 38.635$ and exhibits a good agreement with the interpolation formula proposed by Gaund and Domb [11] (see the plot of both curves in figure 3 of [10]).

(3) It is interesting to note that the obtained equation of states also describes metastable states in the low-temperature phase with oppositely directed external magnetic field and magnetization. Really, if $t < 0$, the equation (12) has appropriate solution $\varphi(p; y_s)$ in the case of negative y_s, at least for small enough values of $|y_s|$. Hence, one can continue the scaling function $h(x_s)$ to the left from the point $x_s = -1$, where $h(x_s)$ changes the sign, down to the point $x_s = x_n$, $x_n < -1$, where the system becomes locally unstable: $\partial H/\partial M|_{x_s=x_n} = 0$.

(4) Let us describe the region in the (H, t)-plane, where the equation of state has the scaling form (10).

The equation (9) becomes invariant with respect to the scaling transformation $\Omega_0 \to \lambda\Omega_0$, $p \to \lambda p$, $HM \to \lambda^2 HM$ in the low-momentum region $|p| \ll p_m$, $p_m = \sqrt{|\Omega_0|/s}$, where the term sp^2 in (9) can be omitted. In this low-momentum region the solution $\varphi(p)$ of equation (9) varies essentially on the scale $p \lesssim \xi^{-1}$ (ξ is the correlation length, see figure 3) approaching then the value $(-\pi/4)$:

$$\varphi(p) \cong -\frac{\pi}{4} + \frac{\Omega_0}{2\Gamma p} + \frac{HM}{2\Gamma p^2}. \tag{22}$$

$$p \gg \xi^{-1}.$$

Estimate from (22) the inverse correlation length:

$$\xi^{-1} \simeq \max\left\{\frac{|\Omega_0|}{2\Gamma}, \left(\frac{HM}{2\Gamma}\right)^{1/2}\right\}. \tag{23}$$

In the well defined scaling region the cut-off momentum p_{m} is to be much larger then the inverse correlation length (23). Thus, we have two inequalities giving the bound of the scaling region:

$$|t| \ll \frac{\Gamma^2}{\varkappa s} \tag{24a}$$

$$HM \ll 2\Gamma|\Omega_0|/s. \tag{24b}$$

Rewrite the latter inequality as

$$x_s^2 \gg \frac{h(0)s\varkappa}{4\Gamma^2}|t|. \tag{25}$$

It is rather surprising that in every neighbourhood of the critical point $t = 0$, $H = 0$ we find states (H, T), where the scaling invariance breaks down.

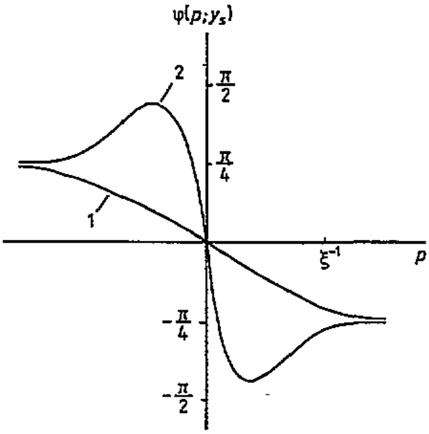

Figure 3. Solutions $\varphi(p)$ of the equation (9) in the low-momentum region $|p| \ll \sqrt{|\Omega_0|/s}$ in the low-temperature phase $\Omega_0 > 0$ (1), and in the high temperature phase $\Omega_0 < 0$ (2).

In conclusion, we obtained representation for the scaling function of the fermionic model which is equivalent in the critical region to the two-dimensional Ising model in uniform magnetic field. The evaluation procedure is not absolutely rigorous, being based on the assumption (6a) of the BCS-like structure of the ground state. However, the above approximation appears to be adequate to the critical region, and it is quite possible, that the derived representation (12), (14), (19) of the scaling function is the exact one.

This work is supported by the Fund of Fundamental Investigations of Republic of Belarus.

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